# Modal Reasoning

Notes by R.J. Buehler
Based on lectures by W. Holliday

January 17, 2014

# Contents

# 1 Flavors of Modality

| Epistemic | It is possible for all they knew that... |
| | It is known by the police that... |

| Temporal | It will sometime be that... |
| | It will always be that ... |

| Alethic | It could have been that... |
| | It is necessary that ... |

| Deontic | It is permissible that ... |
| | It is obligatory that ... |

---

**Example 1** (**Aristotle's Sea Battle Tomorrow**[1]).
*A general is contemplating whether or not to give an order to attack. The general reasons as follows:*

1. *If I give the order to attack, then–necessarily–there will be a sea battle tomorrow.*

2. *If not, then–necessarily–there will not be one.*

3. *Now, I give the order or I do not.*

4. *Hence, either it is necessary that there is a sea battle tomorrow or it is necessary that none occurs.*

*So, why should the the general bother giving the order? There are two possible formalizations of this argument corresponding to different readings of 'If A, then–necessarily–B':*

$$
\begin{array}{ll}
A \to \Box B & \Box(A \to B) \\
\neg A \to \Box \neg B & \Box(\neg A \to \neg B) \\
\underline{A \vee \neg A} & \underline{A \vee \neg A} \\
\Box B \vee \Box \neg B & \Box B \vee \Box \neg B
\end{array}
$$

*Are these two formalizations the same? Are they valid?*

---

**Example 2** (**The Gentle Murder Paradox**). *Suppose that Jones murders Smith. Accepting the principle that 'If Jones murders Smith, Jones ought to murder Smith gently', we can argue that Jones ought to murder Smith as follows:*

1. *Jones murders Smith ($M$)*

2. *If Jones murders Smith, then Jones ought to murder Smith gently ($M \to \Box G$)*

3. *Jones ought to murder Smith gently ($\Box G$)*

4. *If Jones murders Smith gently, the Jones murders Smith ($G \to M$)*

5. *If Jones ought to murder Smith gently, then Jones our to murder Smith ($\Box G \to \Box M$)*

6. *Jones ought to murder Smith ($\Box M$)*

*Is this argument valid?*[1]

> **Example 3.**
> *Consider, for example, alethic modality.*
>
> | | |
> |---|---|
> | $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ | *Valid* |
> | $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$ | *Invalid* |
> | $\Box(p \vee q) \to (\Box p \vee \Box q)$ | *Invalid* |
> | $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ | *Valid* |

The symmetry in the example above is a result of a relationship between $\Box$ and $\Diamond$ first suggested by Aristotle: $\Box p \equiv \neg \Diamond \neg p$.

> **Example 4.**
> *Consider an expanded language as follows:*
>
> | | |
> |---|---|
> | $[F]$ | *It will always be the case* |
> | $\langle F \rangle$ | *It will sometimes be the case* |
> | $[P]$ | *It was always the case* |
> | $\langle P \rangle$ | *It was sometimes the case that* |
>
> *These additional operators allow for the representation of tenses, consider: If it's going to rain in the future, then it's necessarily the case that it's going to rain in the future or it's raining now or it rained in the past.*
>
> $$\langle F \rangle r \to [F](\langle F \rangle r \vee r \vee \langle P \rangle r)$$

Various principles are endorsed in various contexts; for example,

$$\Diamond p \to \Box \Diamond p$$

In the context of the alethic modalities, this principle has often been suggested and corresponds to euclidean closure amongst the possible worlds. In other contexts, say temporal, this principle is obviously false. Thus, while there is a great deal of similarity amongst modalities, they are not all equivalent.

Even within the same modality, however, we may introduce another level of complexity by allowing multiple agents and subscripts below alethic operators for them.

$$\neg(\Box_b r \vee \Box_b \neg r) \wedge \Box_b(\Box_c r \vee \Box_c \neg r)$$

# 2 Basic Language and Semantics[2]

## 2.1 Syntax of Modal Propositional Logic

**Definition: Basic Modal Language**

$$\text{Atoms} := p, q, r, \ldots, \top, \bot$$

And now the remaining sentences are defined inductively by:

$$\phi ::= \text{Atoms} \mid \neg\phi \mid (\phi \wedge \psi) \mid (\phi \vee \psi) \mid (\phi \to \psi) \mid \Diamond\phi \mid \Box\phi$$

Note that greek letters–e.g. $\phi$ and $\psi$ above–denote arbitrary propositions.

One of the core components of modal logic is the duality of the $\Diamond$ and $\Box$ operators–like that between $\exists$ and $\forall$. In particular, the following two principles are intuitively valid:

$$\diamond\phi \leftrightarrow \neg\square\neg\phi$$

$$\square\phi \leftrightarrow \neg\diamond\neg\phi$$

for all the modal interpretations: always and sometimes, necesssarily and possibly, obligation and permission, already and not yet, etc. In the end, this means that we may take either $\square$ or $\diamond$ as primitive.

## 2.2   Important Modal Patterns

### 2.2.1   Distinguishing the Scopes of Modal Operators

Modal operators often give rise to scope ambiguities. For example, the sentence 'If you do $p$, then you must do $q$' has two nonequivalent readings:

$$p \to \square q \qquad \text{Narrow Scope}$$
$$\square(p \to q) \qquad \text{Wide Scope}$$

Another similar ambiguity is rather infamous in modal logic and results from the interaction between modal operators and quantifiers. Consider the sentence 'I know that someone appreciates me'. Again, there are two distinct readings available:

$$\square\exists x A(x,m) \qquad \text{De Dicto}$$
$$\exists x \square A(x,m) \qquad \text{De Re}$$

### 2.2.2   Iteration of Modal Operators

Unlike natural language (mostly), modal logics allow the stacking of modal operators as in the hotly debated 'positive' and 'negative' introspection principles:

$$\square p \to \square\square p \qquad \text{Positive Introspection}$$
$$\neg\square p \to \square\neg\square p \qquad \text{Negative Introspection}$$

> **Definition: Modal Depth**
> The modal depth of a formula $\phi$ or $md(\phi)$ is the maximal length of a nested sequence of modal operators. This can be defined by the following recursion:

1. $md(p) = 0$

2. $md(\neg\phi) = md(\phi)$

3. $md(\phi \wedge \psi) = md(\phi \vee \psi) = md(\phi \to \psi) = max(md(\phi), md(\psi))$

4. $md(\diamond\phi) = md(\square\phi) = md(\phi) + 1$

## 2.3   Model Theory for Modal Propositional Logic

Our modal logic languages will be interpreted over graph-like structures:

> **Definition: Possible Worlds Models**
> A *possible worlds model* is a triple $\mathcal{M} = (W, R, V)$ of a non-empty set of possible worlds $W$, a binary accessibility relation $R$ between worlds, and a valuation map $V : \text{Atoms} \times W \to \{0, 1\}$ assigning truth values (0,1) to proposition, world pairs, e.g. $V(pi, w) = 1$.

> **Definition: Pointed Model**
> A possible worlds model $\mathcal{M}$ paired with a 'current world' or 'vantage point' $w \in W$, i.e. $(\mathcal{M}, w)$.

### 2.3.1 Truth in a Model

A modal formula $\phi$ is true at world $s$ in model $\mathcal{M} = (W, R, V)$ written $\mathcal{M}, s \models \phi$, in virtue of the following recursive definition:

$$
\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & V(p, s) = 1 \\
\mathcal{M}, s \models \neg\phi & \text{iff} & \text{not } \mathcal{M}, s \models \phi \\
\mathcal{M}, s \models \phi \wedge \psi & \text{iff} & \mathcal{M}, s \models \phi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \Box\phi & \text{iff} & \text{for all } t \text{ with } sRt, \ \mathcal{M}, t \models \phi \\
\mathcal{M}, s \models \Diamond\phi & \text{iff} & \text{for some } t \text{ with } sRt, \ \mathcal{M}, t \models \phi
\end{array}
$$

Remember that, with stacked modal operators, we move from the outside in; for example,

---

**Example 5.**

$$\mathcal{M}, w_1 \models \Box\Diamond\Diamond p$$

*Interpreted as, 'From $w_1$, all reachable worlds have $\Diamond\Diamond p$ as true. From each of these worlds, there is a world where $\Diamond p$ is true. Finally, from each of these worlds, there is an accessible world where p.'*

---

As a final note, it's important to be aware of so called 'dead-end' worlds that don't access any worlds; these worlds make $\Box\phi$ trivially true for all $\phi$.

# 3 Model-Theoretic Validity

> **Definition: Model-Theoretic Validity**
> A modal formula $\phi$ is valid, written as '$\models \phi$' if and only if $\mathcal{M}, s \models \phi$ for all models and all worlds.

# 4 Expressive Power and Invariance[2]

## 4.1 Invariance and Expressive Power

The expressive power of any language can be measured by its ability to distinguish between two situations or–equivalently–the situations it considers to be indistinguishable. To capture the expressive power of a language, it's necessary to to find an appropriate structural invariance between models. In first-order logic, the basic invariance is mathematical isomorphism; that is, a structure-preserving bijection between models that leaves all basic properties and relations of objects unchanged. In logic, there is a key trade off between the expressiveness of a language and its computational complexity; expressiveness has a price.

## 4.2    Definability

It's not hard to see that, given the structures supplied so far, it's possible to generate the set of worlds in a model $\mathcal{M}$ where an arbitrary formula $\phi$ is true, written $(\phi)^{\mathcal{M}}$. Determining whether an arbitrary set has a formula that picks it out is a far greater challenge.

> **Definition: Definable Subset**
> Let $\mathcal{M} = \langle W, R, V \rangle$ be a modal model. A set $X \subseteq W$ is definable in $\mathcal{M}$ if
> $X = (\phi)^{\mathcal{M}} = \{w \in W | \mathcal{M}, w \models \phi\}$ for some modal formula $\phi$.

> **Definition: Modal Equivalence**
> Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two modal models. We say $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ are modally equivalent
> provided that for all modal formula $\phi$, $\mathcal{M}_1, w_1 \models \phi$ if and only if $\mathcal{M}_2, w_2 \models \phi$, notated
> $\mathcal{M}_1, w_1 \models \phi \leftrightsquigarrow \mathcal{M}_2, w_2 \models \phi$.
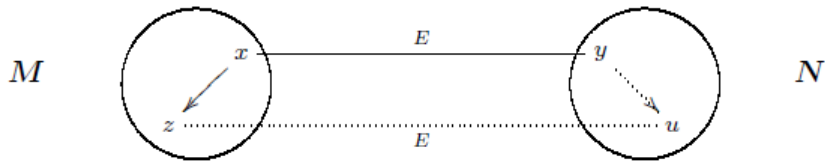
## 4.3    Bisimulation

The model-theoretic invariant for the basic modal logic presented above is 'modal bisimulation':

> **Definition: Modal Bisimulation**
> A bisimulation is a binary relation $E$ between the worlds of two pointed models $\mathcal{M}, s$ and $\mathcal{N}, t$
> such that $sEt$ and also, for any worlds $x, y$ whenever $xEy$, then
>
>   1. Atomic Harmony: $x, y$ verify the same proposition letters
>
>   2. (a) Zig: if $xRz$ in $\mathcal{M}$, then there exists $u$ in $\mathcal{N}$ with $yRu$ and $zEu$.
>      (b) Zag: if $yRu$ in $\mathcal{N}$, then there exists $z$ in $\mathcal{M}$ with $xRz$ and $zEu$.
>
> We notate this as $\mathcal{M}, s \leftrightarrow \mathcal{N}, t$.



If there exists a bisimulation (there could be more than one) between two models, they are said to be bisimilar. If there is a relation between two worlds in a bisimulation, we say the the pointed models from those worlds are bisimilar. Note further that the union of bisimulations is also a bisimulation, and that if we take the union of all possible bisimulations we get a maximal bisimulation which is an equivalence relation.

Bisimulations have two major uses; we consider tree unraveling first, then model contraction.

> **Definition: Tree Unraveling**
> Every modal $\mathcal{M}, s$ has a bisimulation with a rooted tree-like model constructed as follows. The
> worlds in the tree unraveling are all finite paths of worlds in $\mathcal{M}$ starting with $s$ and passing to
> $R$-successors at each step. One path has another path accessible if the second is one step longer
> than the first. The valuation on paths is copied from that on their last nodes.

> **Definition: Model Contraction**
> First observe that any model $\mathcal{M}$ has bisimulations with respect to itself, for instance, the identity relation. Also, given any family of bisimulations $\{E_i\}_{i \in I}$ between two models $\mathcal{M}, \mathcal{N}$, it is easy to see that their set-theoretic union $\bigcup_i \in IE_i$ is again a bisimulation: the latter is called the largest bisimulation. Now...pg.27...

## 4.4 Modal Invariance

The key result, historically, for bisimulation is the following invariance lemma:

▶ **Lemma.**

> For any bisimulation $E$ between models $\mathcal{M}$ and $\mathcal{N}$ and any two worlds $x, y$ with $xEy$:
>
> $$\mathcal{M}, x \models \phi \text{ if and only if } \mathcal{N}, y \models \phi \text{ for all modal formulas } \phi$$
>
> That is, the pointed models $\mathcal{M}, x$ and $\mathcal{N}, y$ are modally equivalent or $\mathcal{M}, x \leftrightsquigarrow \mathcal{N}, y$.

The proof of this result is available on pg. 29 of van Bentham's *Modal Logic for Open Minds*. It's now possible to rigorously show that some properties are undefinable in particular modal languages; the primary means of doing so is to provide two pointed models–$\mathcal{M}, s$ and $\mathcal{N}, t$ –and a bisimulation between them wherein the property in question holds at $s$, but not at the $E$-connected world $t$.

## 4.5 Some Modal Model Theory

In some cases, it's possible to expand the statement above by reversing the conditional:

▶ **Proposition.**

> Let worlds $s, t$ satisfy the same modal formulas in two finite models $\mathcal{M}, \mathcal{N}$. Then there exists a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ connecting $s$ to $t$.

Unfortunately, this does not hold for infinite models. To make this leap, we must extend our modal language to an infinitary version, allowing arbitrary infinite conjunctions and disjunctions.

▶ **Theorem 4.1.**

> The following are equivalent for any two modal models $\mathcal{M}, s$ and $\mathcal{N}, t$:
>
> 1. $s$ and $t$ satisfy the same infinitary modal formulas,
>
> 2. there is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ connecting $s$ with $t$.

## 4.6 Submodels

> **Definition: Submodel**

> **Definition: Generated Submodel**
> Given a model $\mathcal{M}$ and $w \in W$, the submodel generated from $w$ si the submodel $\mathcal{M}'$ such that
>
> - $W'$

# 5 Validity and Decidability

## 5.1 The Minimal Modal Logic

> **Definition: Valid**
> A modal formula is valid if it is true in all possible worlds in all models. The valid formulas form the minimal modal logic.

Decidability is of a great deal of interest with logical systems following to both sides; for example, propositional logic is decidable while first-order logic is not.

▶ **Theorem 5.1.**
*The minimal modal logic is decidable.*

There are a number of ways to prove this; we consider several here that give insights into the structure of the minimal modal logic.

## 5.2 The Finite Model Property

Basic modal logic satisfies the finite model property or FMP:

▶ **Theorem 5.2.**
*Every satisfiable modal formula has a finite model.*

The finite model property doesn't itself give decidability; we may still have to check all finite models– all infinitely many of them. A strengthened version of the finite modal property does, however. In particular, if it's possible to find an effective upper bound on the size of a verifying model in terms of a given formula $\phi$, we say a logic has the *effective finite model property*.

▶ **Theorem 5.3.**
*Modal logic has the effective finite model property.*

A proof of this can be found on pg. 38 of Bentham's *Modal Logic for Open Minds*; the proof given implicitly uses another property of modal logic:

> **Definition: Finite Depth Property**
> For any model $\mathcal{M}, s$ and modal formula $\phi$, $\mathcal{M}, s \models \phi$ if and only if $\mathcal{M}|k, s \models \phi$, where $\mathcal{M}|k, s$ is the submodel of $\mathcal{M}$ whose domain consists of $s$ plus all worlds reachable from it in at most $k$ successor steps, with $k$ with modal depth of $\phi$.

### 5.2.1 Filtrated Models

The finite depth property is reminiscent of the general method of filtration, outlined below.

> **Definition: Filtrated Model**
> Consider any model $\mathcal{M}$, and take any modal formula $\phi$. The filtrated model $\mathcal{M}|\phi$ arises as follows. Set $w \sim v$ if worlds $w, v$ agree on the truth value of each sub-formula of $\phi$. Take the equivalence classes of $w^\sim$ of this relation as the new worlds. For accessibility, set $w^\sim R v^\sim$ if and only if there are worlds $s \sim w$ and $t \sim v$ with $sRt$. Finally, for the valuation, set $w^\sim \models p$ if and only if $w \models p$.

## 5.3 Modal Decomposition

A modal sequent is an expression of the form:

$$\phi_1, \phi_2, \ldots, \phi_n \Rightarrow \psi_1, \psi_2, \ldots, \psi_k$$

and is interpreted in our formal language as

$$\phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi_1 \vee \cdots \vee \psi_k$$

Decomposition Rules

1. a sequent with only atoms is valid if and only if the same atom appears on both sides.

2. $\mathcal{A}, \neg\phi \Rightarrow \mathcal{B}$ is valid if and only is $\mathcal{A} \Rightarrow \mathcal{B}, \phi$ is valid.

3. $\mathcal{A} \Rightarrow \mathcal{B}, \neg\phi$ is valid if and only is $\mathcal{A}, \phi \Rightarrow \mathcal{B}$ is valid

4.

5.

6. $p, \Diamond\phi_1, \ldots, \Diamond\phi_k \Rightarrow \Diamond\psi_1, \ldots, \Diamond\psi_m, q$ is valid if and only if either:

    (a) $p$ and $q$ overlap, or
    (b) for some $i \leq k, \phi_i \Rightarrow \psi_1, \ldots, \psi_m$ is valid.

---

**Example 6.**

---

A proof of the validity of the method given can be found on page 41 of [2] .

## 5.4 Semantic Tableau

# 6 Axioms, Proofs, and Completeness

## 6.1 Proof Theoretic Validity

To show that something is a validity, the algorithms described above, while feasible, aren't particularly natural or intuitive. We consider, then, a proof theoretic presentation of the minimal modal logic.

> **Definition: Minimal Modal Logic**
> The minimal modal logic $K$ is the Hilbert-style proof system with the following axioms:
>
> 1. all tautologies from propositional logic (with modal operators)
>
> 2. modal distribution or the 'K' axiom: $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$,
>
> 3. definition of $\Diamond\phi$ as $\neg\Box\neg\phi$,
>
> 4. the rule Modus Ponens,
>
> 5. and a rule of necessitation: "if $\phi$ is provable, then so is $\Box\phi$".

Proofs are finite sequences of formulas, each of them either ($i$) an instance of an axiom or ($ii$) the result of applying a derivation rule to preceding formulas. A formula $\phi$ is provable, symbolized '$\vdash \phi$', if there exists a proof ending in $\phi$. If we wish to indicate the logic being used, we write the name of the system as a subscript, e.g. $\vdash_K \phi$.

## 6.2 Soundness and Completeness

▶ **Theorem 6.1** (Weak Soundness and Completeness).

*For all modal formulas $\phi$, $\vdash_K \phi$ if and only if $\models \phi$*

In lieu of proving the full result, we give an overview of key parts of the proof. Weak soundness is easy to prove by inspection. To achieve the completeness result, we argue by contraposition and utilize a cover argument, utilizing the following concept:

> **Definition: Consistency**
> A set $\Sigma$ of formulas is consistent if for no finite conjunction $\sigma$ of formulas from $\Sigma$, the negation $\neg\sigma$ is provable in the logic $K$.

Consistent sets have useful properties used here without proof; in particular, if $\phi$ is not derivable, then the set $\{\neg\phi\}$ is consistent. We now prove that any consistent set of formulas $\Sigma$ has a satisfying model.

### 6.2.1 Maximally Consistent Sets

> **Definition: Maximally Consistent Set**
> A consistent set of formulas which has no consistent proper extensions.

Any consistent set of formulas is contained in a maximally consistent set–a statement itself provable from general set-theoretic principles (Zorn's Lemma). Maximally consistent sets have a number of nice properties; in particular, if $\Sigma$ is a maximally consistent set:

(i) $\neg\phi \in \Sigma$ if and only if not $\phi \in \Sigma$

(ii) $\phi \wedge \psi \in \Sigma$ if and only if $\phi \in \Sigma$ and $\psi \in \Sigma$

It follows easily that they are also closed under $K$-derivable formulas.

(iii) $\Diamond\phi \in \Sigma$ if and only if there is some $\Delta$ with $\Sigma R\Delta$ and $\phi \in \Delta$

The proof is trivial from right to left. The reverse direction is the first place to require the modal properties of $K$. Consider the set $\Gamma = \{\phi\} \cup \{\alpha|\Box\alpha \in \Sigma\}$; by the earlier definition of accessibility, any maximally consistent set containing this will be an $R$-successor of $\Sigma$. It's not difficult to prove that $\Gamma$ is consistent (see [2] pg. 55).

### 6.2.2 Henkin Model

Now we define a model $\mathcal{M} = (W, R, V)$ as follows:

> **Definition: Canonical Model**
> The worlds $W$ are all maximally consistent sets, the accessibility relation is the above defined relation $R$, and for the propositional valuation $V$, we set $\Sigma \in V(p)$ if and only if $p \in \Sigma$.

Everything is now in place for a final lemma, the so called 'truth lemma':

▶ **Lemma** (Truth Lemma).

*For each maximally consistent set $\Sigma$ and each modal formula $\phi$,*

$$\mathcal{M}, \Sigma \models \phi \text{ if and only if } \phi \in \Sigma$$

It's worth noting that the lemma above actually establishes a stronger result (strong completeness) than that which we set out to prove (completeness). Another interesting corollary is that all consistent sets can be made true in one and the same model! It is a characteristic fact about the modal language that makes the Henkin model 'largest' or 'universal' among all models.

# 7 Correspondence Theory

## 7.1 Famous Axioms

| | |
|---|---|
| T | $\Box\phi \to \phi$ |
| D | $\Box\phi \to \Diamond\phi$ |
| 4 | $\Box\phi \to \Box\Box\phi$ |
| 5 | $\Diamond\phi \to \Box\Diamond\phi$ |
| B | $\phi \to \Box\Diamond\phi$ |

We name systems beyond $K$ by the listing of the additional axioms to the right of $K$; common examples are the system S4 as KT4 and S5 as KTR5. Why these names/letters??????????????

## 7.2 The Connection between Axioms and Frames

Kanger-Kripke semantics are so attractive in large part due to the correspondence between modal axioms and the modal accesibility relation $R$; some of the best known of these are:

| | | |
|---|---|---|
| $T$-axiom | $\Box p \to p$ | reflexivity |
| $K4$-axiom | $\Box p \to \Box\Box p$ | transitivity |
| $S5$-axiom | $\Diamond\Box p \to p$ | symmetry |
| $D$-axiom | $\Box\phi \to \Diamond\phi$ | serial ($\forall w \in W,\ \exists v \in W : wRv$) |
| 5 | $\Diamond\phi \to \Box\Diamond\phi$ | Euclidean (($wRv$ and $wRu$)$\to uRv$) |
| | $\Box(\Box\phi \to \phi)$ | Shift Reflexivity ($\forall w \forall v\ wRv \to vRv$) |

We introduce the notion of truth in a frame to make this correspondence explicit:

> **Definition: Truth in a Frame**
> Let $\mathcal{F} = (W, R)$ be a frame, $w$ a world. We write $\mathcal{F}, w \models \phi$ if and only if $\mathcal{F}, V, w \models \phi$ for all valuations $V$. Additionally, we write $\mathcal{F} \models \phi$ if and only if $\phi$ is true at all worlds $w$ in $\mathcal{F}$.

Proof of correspondence, pg. 102; pg. 11 Pacuit is better

# 8 Non-normal Modal Logics

> **Definition: Normal Modal Logic**
> Any logic which includes $K$ and is closed under uniform substitution (i.e. if $\phi$ is a theorem, so is the result $\phi'$ of uniformly replacing atomic symbols in $\phi$ by any formulas.)

This is the sense in which $K$ is the minimal normal modal logic.

> **Definition: Neighborhood Model**
> A tuple $\mathcal{M} = \rangle\langle$

# References

[1] Eric Pacuit. Notes on Modal Logic. 2009. ai.stanford.edu/~epacuit/classes/ml-notes.pdf.

[2] Johan van Benthem. *Modal Logic for Open Minds*. CSLI Publications, 2010.